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Abstract

In this paper we show that the complex Burgers and the Kundu–Eckhaus equations are related by a Miura transformation. We use this relation to discretize the Kundu–Eckhaus equation.

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1. Introduction

The Burgers equation is the simplest partial differential equation that combines nonlinear wave propagation with diffusive effects. As such, it has been widely applied to the modelling of physical processes such as sedimentation, shock propagation in gaseous flow, turbulence in fluids and road traffic [4, 13]. The name of the Burgers equation was introduced by Hopf as a reference to the results of Burgers [4], but, in fact, the Burgers equation can be found already in a work by Bateman published in 1915 [3]. An explicit solution of the Cauchy problem on the infinite line for the Burgers equation may be obtained by linearizing Hopf–Cole transform, introduced independently by Hopf and Cole in 1950 [6, 11]. This transformation is already contained in an article by Florin [8] published in 1948 and implicitly in a book by Forsyth [9] published in 1906.

Kundu [12] and Eckhaus [5, 7] independently derived in 1984–1985 what can now be called the Kundu–Eckhaus equation as a linearizable form of the nonlinear Schrödinger equation.

Here in the following we show that the Kundu–Eckhaus equation can be related to the complex Burgers equation by a Miura transformation. Then taking into account this relation we are able to discretize the Kundu–Eckhaus equation using the discretization procedure introduced for the Burgers equation.

In section 2 we present the linearization procedure for both the Burgers and the Kundu–Eckhaus equations and use it to derive the Miura transformation which relates them. Then in section 3 we use the standard discretization via Bäcklund transformation, used to construct the discrete Burgers equation, to discretize the Kundu–Eckhaus equation.

2. The complex Burgers equation and the Kundu–Eckhaus equation

Let us consider a complex extension of the Burgers:

$$iu_t + u_{xx} + 2uu_x = 0 \tag{1}$$

where $u(x, t)$ is a complex field.   Equation (1), introducing the standard Hopf–Cole transformation

$$u(x, t) = \frac{\phi_x}{\phi}, \tag{2}$$

reduces to the time-dependent free Schrödinger equation

$$i\phi_t + \phi_{xx} = 0, \tag{3}$$

provided that the time evolution of the function $\phi(x, t)$ satisfies the linear ordinary differential equation

$$\phi_t = i(u_x + u^2)\phi|_{x=a}, \tag{4}$$

where a is an arbitrary value of the x variable at which all the functions involved are well defined. The solution of equation (4) gives

$$\phi(a, t) = \phi(a, b) \, e^{i \int_b^t (u_x + u^2)|_{x=a} dt'}, \tag{5}$$

where b is an arbitrary value of the t variable at which all the functions involved are well defined. Consequently, as is well known, the inverse of the Hopf–Cole transformation (2) reads

$$\phi(x, t) = \phi(a, t) \, e^{\int_a^x u \, dx'}. \tag{6}$$

From equation (1) we derive that an asymptotically bounded solution will be such that $\lim_{x \to \infty} u(x, t) = u(-\infty, t) = u_0$, where u_0 is a finite constant. In this case the above formula can be replaced by

$$\phi(x, t) = \alpha(t) \, e^{u_0 x} \, e^{\int_{-\infty}^x (u - u_0) dx'}, \tag{7}$$

where $\alpha(t)$ is a t-dependent function. The function (7) will satisfy the linear Schrödinger equation (3) if

$$\alpha(t) = \alpha_0 \, e^{iu_0^2 t}, \tag{8}$$

where α_0 is a constant. When $u_0 = 0$, α_0 equals the asymptotic value of $\phi(x, t)$.

A less known linearizable (or C-integrable) equation is the Kundu–Eckhaus equation

$$i\psi_t + \psi_{xx} + 2\psi |\psi|_x^2 + \psi |\psi|^4 = 0. \tag{9}$$

This is a nonlinear Schrödinger-type equation that also linearizes to the free linear Schrödinger equation (3). As the well-known nonlinear Schrödinger equation, it is a universal model equation and, as such, it appears in many applications. For example, it has been obtained in the study of the instabilities of plane solitons associated with the Kadomtsev–Petviashvili equation [10].

The Kundu–Eckhaus equation linearizes to the time-dependent free Schrödinger equation (3) through the following procedure. Let us define the complex function

$$\phi = \sqrt{2\Phi}\psi, \tag{10}$$

where the real function Φ is related to ϕ by the following overdetermined system of equations

$$\Phi_x = |\phi|^2, \tag{11a}$$

$$\Phi_t = \mathrm{i}[\bar{\phi}\phi_x - \phi\bar{\phi}_x], \tag{11b}$$

where by a bar we indicate the complex conjugate. The Kundu–Eckhaus equation (9) is obtained by inserting equation (10) into equation (3) and taking into account equations (11). The compatibility of equations (11) is identically satisfied on the solutions of equation (3). Solving equations (11), we get

$$\Phi = \int_a^x |\phi|^2 \, \mathrm{d}x' + \frac{1}{2}\rho(t), \tag{12}$$

$$\rho(t) = 2\mathrm{i} \int_b^t (\bar{\phi}\phi_x - \phi\bar{\phi}_x)|_{x=a} \, \mathrm{d}t' + \rho_0, \tag{13}$$

where ρ_0 is an arbitrary real constant. Then equation (10) can be written as

$$\psi = \frac{\phi}{\left[2\int_a^x |\phi|^2 \, \mathrm{d}x' + \rho(t)\right]^{1/2}}. \tag{14}$$

Equations (13) and (14) can be inverted giving

$$\phi = \sqrt{\rho(t)} \, \mathrm{e}^{\int_a^x |\psi|^2 \mathrm{d}x'} \psi, \tag{15a}$$

$$\rho(t) = \rho_0 \, \mathrm{e}^{2\mathrm{i} \int_b^t (\bar{\psi}\psi_x - \psi\bar{\psi}_x)|_{x=a} \mathrm{d}t'}. \tag{15b}$$

We see that, if we set the lower extremum of integrations $a = -\infty$, both ϕ and ψ must go to zero as $x \to -\infty$. Let us point out that we must have $\rho(t)/\rho_0 \geqslant 0$ as one can deduce from equation (15b). Moreover, as a direct consequence of equation (10) we get the following relation between ϕ and ψ:

$$\phi\bar{\psi} = \bar{\phi}\psi. \tag{16}$$

By differentiating equation (10) with respect to x and using equations (11a), (16), we get the following differential equation relating ϕ and ψ:

$$\phi_x = \left(\frac{\psi_x}{\psi} + |\psi|^2\right)\phi. \tag{17}$$

By comparing equations (2), (17) we obtain a Miura transformation between the function ψ and the function u satisfying the complex Burgers equation (1):

$$u = \frac{\psi_x}{\psi} + |\psi|^2. \tag{18}$$

The inversion of equation (18) is obtained by combining equations (5), (6) and (14). It reads

$$\psi = \frac{A(t) \, \mathrm{e}^{\int_a^x u \mathrm{d}x'}}{\left[2|A(t)|^2 \int_a^x \mathrm{e}^{\int_a^{x'}(u+\bar{u})\mathrm{d}x''} \mathrm{d}x' + \rho(t)/\rho_0\right]^{1/2}}, \tag{19a}$$

$$\rho(t) = 2\mathrm{i}\rho_0 \int_b^t |A(t)|^2 (u - \bar{u})\Big|_{x=a} \, \mathrm{d}t' + \rho_0, \tag{19b}$$

$$A(t) = \psi(a, b) \, \mathrm{e}^{\mathrm{i} \int_b^t (u_x + u^2)|_{x=a} \mathrm{d}t'}, \tag{19c}$$

where $\psi(a, b) = \phi(a, b)\big/\rho_0^{1/2}$. Naturally the Kundu–Eckhaus equation can also be obtained by introducing the Miura transformation (18) into the complex Burgers equation (1), fixing $\psi(a, t)$ in such a way that it is consistent with equation (19).

A solution of the Kundu–Eckhaus equation is given by equation (14) in terms of a solution of the Schrödinger equation (3). Equation (18) gives the solution of the complex Burgers equation in terms of the solution of the Kundu–Eckhaus equation. Equation (19) provides the solution of the Kundu–Eckhaus equation in terms of the solutions of the complex Burgers equation. So, by solving the linear Schrödinger equation (3) we can get solutions of both the complex Burgers and Kundu–Eckhaus equations.

This constructive procedure to get the Kundu–Eckhaus equation can be discretized and provide the differential difference and difference difference Kundu–Eckhaus equation.

3. Discretizations

Equation (10) is a functional relation and, as such, it is valid when all involved fields, Φ, ϕ and ψ, depend not only on continuous variables but also on discrete variables. Similarly, as a consequence, the same is true for equation (16). What will change when discretizing is the linear equation (3), the overdetermined linear system for Φ and the resulting Burgers and Kundu–Eckhaus equations.

Let us start from the differential difference case when we just discretize the space variable x. In this case we assume as a free linear Schrödinger equation the differential difference equation

$$\mathrm{i}\dot{\phi}_n + \frac{\phi_{n+1} + \phi_{n-1} - 2\phi_n}{h^2} = 0, \tag{20}$$

where h is the lattice spacing and n is the lattice index such that $x = nh$. The general solution of the previous equation is obtained by using the Z-transform and reads

$$\phi_n(t) = \frac{1}{2\mathrm{i}\pi} \oint_C \tilde{\phi}(z)\, \mathrm{e}^{\frac{\mathrm{i}}{h^2}(z+1/z-2)t} z^{n-1}\, \mathrm{d}z, \tag{21}$$

$$\tilde{\phi}(z) = \sum_{n=-\infty}^{+\infty} \phi_n(0) z^{-n}, \tag{22}$$

where C is a counterclockwise circle in the complex z-plane, centred in $z = 0$ and encircling all the poles of $\tilde{\phi}(z)$. This representation of the solution is only valid for z inside the region of convergence of the series in equation (22) (and C must obviously be contained inside this region). By compatibility of equation (20) with the discrete Hopf–Cole transformation

$$\frac{\phi_{n+1} - \phi_n}{h} = u_n \phi_n, \tag{23}$$

we get the discrete complex Burgers

$$\mathrm{i}\dot{u}_n + \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \frac{1}{h}\left[u_n(u_{n+1} - u_n) + \frac{u_{n-1}(u_n - u_{n-1})}{(1 + hu_{n-1})} \right] = 0. \tag{24}$$

The inverse of the discrete Hopf–Cole transformation (23) reads

$$\phi_n = \phi_a \prod_{j=a}^{j=n-1} (1 + hu_j), \qquad n \geqslant a + 1, \tag{25a}$$

$$\phi_n = \frac{\phi_a}{\prod_{j=n}^{j=a-1}(1+hu_j)}, \qquad n \leqslant a-1, \tag{25b}$$

where $\phi_a = \phi_a(t)$ is the function ϕ_n calculated at the arbitrary point $n = a$. For any bounded asymptotically solution of equation (24) $\lim_{n\to-\infty} u_n(t) = u_0$, a finite constant. In this case, the above formulae can be replaced by

$$\phi_n = \alpha(t)\,(1+hu_0)^n \prod_{\gamma=-\infty}^{\gamma=n-1}\left(\frac{1+hu_\gamma}{1+hu_{-\infty}}\right), \tag{26}$$

where $\alpha(t)$ is a t-dependent function. When $u_n(t)$ satisfies the complex Burgers equation (24), ϕ_n, given by equation (25), will satisfy the discrete linear Schrödinger equation (20) if $\phi_n(t)$ satisfies the ordinary differential equation

$$\mathrm{i}\dot{\phi}_n + \frac{1}{h^2}\left[hu_n - 1 + \frac{1}{(1+hu_{n-1})}\right]\phi_n\bigg|_{n=a} = 0. \tag{27}$$

The solution of equation (27) is

$$\phi_a(t) = \phi_a(b)\,\mathrm{e}^{\frac{\mathrm{i}}{h^2}\int_b^t [hu_n - 1 + \frac{1}{(1+hu_{n-1})}]|_{n=a}\,\mathrm{d}t'}, \tag{28}$$

where b is an arbitrary value of the time variable. Otherwise, if the solution is given by formula (26),

$$\alpha(t) = \alpha_0\,\mathrm{e}^{\mathrm{i}\frac{u_0^2 t}{1+hu_0}}, \tag{29}$$

where α_0 is an arbitrary constant that, when $u_0 = 0$, equals $\lim_{n\to-\infty}\phi_n(t)$.

To construct the differential difference Kundu–Eckhaus equation we replace the overdetermined system of equations for $\Phi(x, t)$ by a system for $\Phi_n(t)$, whose compatibility is satisfied on the solutions of the differential difference linear Schrödinger equation (27). We get

$$\Phi_{n+1} - \Phi_n = h|\phi_n|^2, \tag{30a}$$

$$\dot{\Phi}_n = \frac{\mathrm{i}}{h}(\bar{\phi}_{n-1}\phi_n - \bar{\phi}_n\phi_{n-1}). \tag{30b}$$

Solving equations (30), we get that equation (10) becomes

$$\psi_n = \frac{\phi_n}{\left[2h\sum_{j=a}^{j=n-1}|\phi_j|^2 + \rho(t)\right]^{1/2}}, \qquad n \geqslant a+1, \tag{31a}$$

$$\psi_a = \frac{\phi_a}{\sqrt{\rho(t)}}, \tag{31b}$$

$$\psi_n = \frac{\phi_n}{\left[-2h\sum_{j=n}^{j=a-1}|\phi_j|^2 + \rho(t)\right]^{1/2}}, \qquad n \leqslant a-1, \tag{31c}$$

$$\rho(t) = \frac{2\mathrm{i}}{h}\int_b^t (\bar{\phi}_{n-1}\phi_n - \bar{\phi}_n\phi_{n-1})\big|_{n=a}\,\mathrm{d}t' + \rho_0, \tag{31d}$$

where ρ_0 is an arbitrary real constant. Equations (31) can be inverted giving

$$\phi_n = \psi_n\sqrt{\rho(t)}\prod_{j=a}^{j=n-1}(1+2h|\psi_j|^2)^{1/2}, \qquad n \geqslant a+1, \tag{32a}$$

$$\phi_a = \psi_a \sqrt{\rho(t)}, \tag{32b}$$

$$\phi_n = \frac{\psi_n \sqrt{\rho(t)}}{\prod_{j=n}^{j=a-1} (1 + 2h|\psi_j|^2)^{1/2}}, \qquad n \leqslant a - 1, \tag{32c}$$

$$\rho(t) = \rho_0\, e^{\frac{2i}{h} \int_b^t \frac{(\bar{\psi}_{n-1}\psi_n - \bar{\psi}_n \psi_{n-1})}{(1+2h|\psi_{n-1}|^2)^{1/2}} |_{n=a} dt'}. \tag{32d}$$

If $a = -\infty$, both ϕ and ψ must go to zero as $n \to -\infty$. From equation (10), taking into account equations (16), (30a) we get

$$\phi_{n+1} = \left(\frac{\psi_{n+1}}{\psi_n} \sqrt{1 + 2h|\psi_n|^2} \right) \phi_n. \tag{33}$$

By comparing equations (23) and (33) we get

$$u_n = \frac{\psi_{n+1}\sqrt{1 + 2h|\psi_n|^2} - \psi_n}{h\psi_n}, \tag{34}$$

that is the discrete Miura transformation between the function $\psi_n(t)$ and the function $u_n(t)$ satisfying the complex differential difference Burgers equation (24). The inversion of equation (34) is obtained considering equations (3), (28), (31), and is given by

$$\psi_n = \frac{A(t) \prod_{j=a}^{j=n-1} (1 + hu_j)}{\left[2h|A(t)|^2 \sum_{j=a}^{j=n-1} \prod_{k=a}^{k=j-1} |1 + hu_k|^2 + \rho(t)/\rho_0 \right]^{1/2}}, \tag{35a}$$

$$\psi_a = \frac{A(t)}{\sqrt{\rho(t)/\rho_0}}, \tag{35b}$$

$$\psi_n = \frac{A(t) \big/ \prod_{j=n}^{j=a-1} (1 + hu_j)}{\left[-2h|A(t)|^2 \sum_{j=n}^{j=a-1} 1 \big/ \prod_{k=j}^{k=a-1} |1 + hu_k|^2 + \rho(t)/\rho_0 \right]^{1/2}}, \tag{35c}$$

$$\rho(t) = 2i\rho_0 \int_b^t \left| \frac{A(t)}{1 + hu_{n-1}} \right|^2 (u_{n-1} - \bar{u}_{n-1}) \bigg|_{n=a} dt' + \rho_0, \tag{35d}$$

$$A(t) = \psi_a(b)\, e^{\frac{i}{h^2} \int_b^t \left[hu_n - 1 + \frac{1}{(1+hu_{n-1})} \right] \big|_{n=a} dt'}, \tag{35e}$$

where $\psi_a(b) = \phi_a(b) \big/ \rho_0^{1/2}$. Equation (35a) is valid for $n \geqslant a + 1$ while equation (35c) for $n \leqslant a + 1$. If one substitutes equation (34) into the complex Burgers equation (24) fixing $\psi_a(t)$ in a consistent way with equation (35), we get the following differential difference equation for the function $\psi_n(t)$:

$$i\dot{\psi}_n + \frac{1}{h^2} \left[\psi_{n+1}\sqrt{1 + 2h|\psi_n|^2} + \frac{\psi_{n-1}}{\sqrt{1 + 2h|\psi_{n-1}|^2}} - 2\psi_n \right]$$
$$- \frac{1}{h} \frac{\psi_n}{\sqrt{1 + 2h|\psi_{n-1}|^2}} [\psi_n \bar{\psi}_{n-1} - \bar{\psi}_n \psi_{n-1}] = 0, \tag{36}$$

the differential difference Kundu–Eckhaus equation. Equation (36) can also be obtained by inserting equation (10) into equation (20) and taking into account equation (30). Carrying out the continuous limit, when $h \to 0$ and $n \to \infty$ in such a way that $x = nh$ remains finite, we obtain the Kundu–Eckhaus equation (9). Equation (31) provides solutions of the differential difference Kundu–Eckhaus equation in terms of the solutions of the differential difference

linear Schrödinger equation. We can solve the discrete linear Schrödinger equation (20) by separation of variables and we get, for example,

$$\phi_n(t) = \beta \kappa^n \, e^{\frac{i}{h^2}(\kappa + 1/\kappa - 2)t}, \tag{37}$$

where β and κ are arbitrary constants, β complex and κ real with $|\kappa| < 1$. Then, by equation (31a) with $a = -\infty$, we obtain the following solution of the differential difference Kundu–Eckhaus equation (36):

$$\psi_n(t) = \kappa^n \, e^{\frac{i}{h^2}(\kappa + 1/\kappa - 2)t + i\varphi_\beta} \bigg/ \left[2h \frac{k^{2n}}{k^2 - 1} + \frac{\rho_0}{|\beta|^2} \right]^{1/2}, \tag{38}$$

where φ_β is the phase of β. To be able to perform the continuous limit we must replace κ with κ^h. In this way, by performing the limit $h \to 0$, equation (37) provides a solution of the linear Schrödinger equation (3) and equation (38) a solution of the Kundu–Eckhaus equation (9):

$$\phi(x, t) = \beta \kappa^{x + it \ln \kappa}, \tag{39}$$

$$\psi(x, t) = \kappa^{x + it \ln \kappa + i\varphi_\beta} \bigg/ \left[\frac{\kappa^{2x}}{\ln \kappa} + \frac{\rho_0}{|\beta|^2} \right]^{1/2}. \tag{40}$$

The completely discrete equation is obtained by discretizing also the time variable. This is done by introducing a new index m and its spacing τ such that $t = m\tau$. In this case, if we want to preserve the linearity of the Lax pair, we have to introduce the following overdetermined system of equations for the real function $\Phi_{n,m}$:

$$\begin{aligned}
\Phi_{n+1,m} - \Phi_{n,m} &= h|\phi_{n,m}|^2, \\
\Phi_{n,m+1} - \Phi_{n,m} &= \frac{i\tau}{h}(\phi_{n,m}\bar{\phi}_{n-1,m} - \bar{\phi}_{n,m}\phi_{n-1,m}) + \tau\sigma_{n,m}.
\end{aligned} \tag{41}$$

where $\sigma_{n,m}$ is a real function which goes to zero when $\tau \to 0$. The compatibility conditions of equation (41) provide the discrete Schödinger equation

$$\frac{i}{\tau}(\phi_{n,m+1} - \phi_{n,m}) + \frac{1}{h^2}(\phi_{n+1,m} + \phi_{n-1,m} - 2\phi_{n,m}) = 0, \tag{42}$$

if the real function $\sigma_{n,m}$ satisfies the following difference equation:

$$\sigma_{n+1,m} - \sigma_{n,m} = \frac{\tau}{h^2}|\phi_{n+1,m} + \phi_{n-1,m} - 2\phi_{n,m}|^2 \tag{43}$$

with the boundary condition $\sigma_{a,m} = 0$. From equations (10) and (41) we get the following n and m evolution of the function $\phi_{n,m}$:

$$\begin{aligned}
\phi_{n+1,m} &= \left[\frac{\psi_{n+1,m}}{\psi_{n,m}} \sqrt{1 + 2h|\psi_{n,m}|^2} \right] \phi_{n,m}, \\
\phi_{n,m+1} &= \left[\frac{\psi_{n,m+1}}{\psi_{n,m}} \sqrt{1 + 2i\frac{\tau}{h^2} \frac{\psi_{n,m}\bar{\psi}_{n-1,m} - \psi_{n-1,m}\bar{\psi}_{n,m}}{\sqrt{1 + 2h|\psi_{n-1,m}|^2}} + \tau\rho_{n,m}} \right] \phi_{n,m},
\end{aligned} \tag{44}$$

where $\rho_{n,m} = \frac{\sigma_{n,m}}{\Phi_{n,m}}$, taking into account the definition of the function $\sigma_{n,m}$ given by equation (43), satisfies the first-order linear difference equation

$$\begin{aligned}
\rho_{n+1,m} - \frac{1}{1 + 2h|\psi_{n,m}|^2}\rho_{n,m} = {}& \frac{2\tau}{h^3[1 + 2h|\psi_{n,m}|^2]} \bigg| \psi_{n+1,m}\sqrt{1 + 2h|\psi_{n,m}|^2} \\
& + \frac{\psi_{n-1,m}}{\sqrt{1 + 2h|\psi_{n-1,m}|^2}} - 2\psi_{n,m} \bigg|^2 \bigg].
\end{aligned} \tag{45}$$

Introducing the two relations (44) into the discrete heat equation (42) we get the following nonlinear discrete partial difference equation:

$$
\mathrm{i}\left[\psi_{n,m+1}\sqrt{1+2\mathrm{i}\frac{\tau}{h^2}\frac{\psi_{n,m}\bar{\psi}_{n-1,m}-\psi_{n-1,m}\bar{\psi}_{n,m}}{\sqrt{1+2h|\psi_{n-1,m}|^2}}+\tau\rho_{n,m}}-\psi_{n,m}\right]
$$
$$
+\frac{\tau}{h^2}\left[\psi_{n+1,m}\sqrt{1+2h|\psi_{n,m}|^2}+\frac{\psi_{n-1,m}}{\sqrt{1+2h|\psi_{n-1,m}|^2}}-2\psi_{n,m}\right]=0, \qquad (46)
$$

the difference difference Kundu–Eckhaus equation. It is easy to see that in the continuous limit equation (46) goes into the Kundu–Eckhaus equation (9) independently from the continuous limit of the function $\rho_{n,m}$. Equation (45) reduces in the same limit to the linear equation $\rho_x = -2|\psi|^2\rho + 2|\psi|^{10}$. Moreover, it is worthy to note that this completely discrete Kundu–Eckhaus equation, as it is for the well-known nonlinear Schrödinger equation written down by Ablowitz and Ladik [2], is nonlocal and it involves the function $\psi_{n,m}$ at all points of the lattice.

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